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A Note on the Stability Properties of Uniform Temperature Fields

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INTRODUCTION

In this brief note the *stability* and *asymptotic stability* properties of temperature fields θ governed by quasilinear heat equations are established. Results of a similar nature have been established by Bellman [1] and Narasimhan [2] for temperature fields governed by semilinear heat equations.

The author feels that Theorem 1.2 should be of particular interest since it establishes a generalized "maximum principle" when Neumann type boundary data are prescribed for times $t \geqslant 0$.

1. PRELIMINARIES AND STATEMENT OF THE MAIN THEOREMS

In what follows Ω will denote a bounded domain of n dimensional Euclidean point space E_n with smooth boundary $\partial\Omega$ and closure $\bar{\Omega}$. Points of Ω, $\partial\Omega$, and $\bar{\Omega}$ will be denoted by $\mathbf{X}$; the parameter $t \in [0, \infty)$ will represent time. R_n will be an n dimensional vector space under the Euclidean norm and R_1^+ the positive real numbers.

$\theta(\mathbf{X}, t)$ will be the temperature, $\epsilon(\mathbf{X}, t)$ the internal energy per unit volume, and $\mathbf{q}(\mathbf{X}, t)$ the heat conduction vector at the point $\mathbf{X}$ at time t.

We assume the existence of two functions $\hat{\epsilon} : R_1^+ \to R_1^+$ and $\hat{\mathbf{q}}: R_1^+ \times R_n \to R_n$ which determine the internal energy and heat conduction vector through the following constitutive equations:

$$\epsilon(\mathbf{X}, t) = \hat{\epsilon}(\theta(\mathbf{X}, t)), \tag{1.1a}$$

and

$$\mathbf{q}(\mathbf{X}, t) = \hat{\mathbf{q}}(\theta(\mathbf{X}, t), \nabla\theta(\mathbf{X}, t)). \tag{1.1b}$$

We assume that the functions $\hat{\epsilon}$ and $\hat{\mathbf{q}}$ satisfy

ASSUMPTION 1.

(a) $\hat{\epsilon}$ *is* $C^2(R_1^+)$ *and has a derivative* $\hat{\epsilon}'$ *which is positive in* R_1^+.[1]

(b) $\hat{\mathbf{q}}$ *is* $C^1(R_1^+ \times R_n)$ *and satisfies*

$$\hat{\mathbf{q}}(\theta, \mathbf{g}) \cdot \mathbf{g} \leqslant 0 \tag{1.2}$$

for all $(\theta, \mathbf{g})$ *in* $R_1^+ \times R_n$ *with equality holding only if* $\mathbf{g} = 0$.[2]

We say that a continuous function $\hat{r}_{\theta_0} : R_1^+ \to (-\infty, \infty)$ is a "temperature control" for the positive temperature θ_0 if it has the following two properties:

$$(TC\text{-}1) \quad \hat{r}_{\theta_0}(\theta_0) = 0, \qquad \text{and} \tag{1.3}$$

$(TC\text{-}2)$ there exists a number $r_0 > 0$ such that

$$\hat{r}_{\theta_0}(\theta)(\theta - \theta_0) \geqslant r_0(\theta - \theta_0)^2 \tag{1.4}$$

for all $\theta > 0$.

We shall be interested in the qualitative properties of "smooth" temperature fields θ which satisfy the balance of energy:

$$\hat{\epsilon}'(\theta)\frac{\partial \theta}{\partial t} + \hat{r}_{\theta_0}(\theta) = -\operatorname{div} \hat{\mathbf{q}}(\theta, \nabla\theta), \qquad (\mathbf{X}, t) \in \Omega \times (0, \infty), \tag{1.5}$$

where $\hat{r}_{\theta_0}$ is a temperature control for the temperature θ_0.

Our main results for such temperature fields are contained in the following theorems.

THEOREM 1. *Let* $\hat{\epsilon}$ *and* $\hat{\mathbf{q}}$ *satisfy Assumption* 1.1 *and let* f *be any* $C^2(\bar{\Omega})$ *function which satisfies*

$$f(\mathbf{X}) = \theta_0, \qquad \mathbf{X} \in \partial\Omega, \tag{1.6a}$$

and

$$0 < f(\mathbf{X}) < 2\theta_0, \qquad \mathbf{X} \in \bar{\Omega}. \tag{1.6b}$$

If θ *is any* $C^2(\bar{\Omega} \times [0, \infty))$ *function which satisfies* (1.5), *the initial condition that* $\theta(\mathbf{X}, 0) = f(\mathbf{X})$ *for all* $\mathbf{X} \in \bar{\Omega}$, *and the time-independent boundary condition that* $\theta(\mathbf{X}, t) = \theta_0$ *for all* $(\mathbf{X}, t) \in \partial\Omega \times (0, \infty)$, *then, for all* $t \geqslant 0$,

$$\sup_{\mathbf{X} \in \bar{\Omega}} | \theta(\mathbf{X}, t) - \theta_0 | \leqslant V(f, \theta_0)\, e^{-Kt}, \tag{1.7}$$

where $V(f, \theta_0) \overset{\text{DEF}}{=} \sup_{\mathbf{X} \in \bar{\Omega}} | f(\mathbf{X}) - \theta_0 |$ *and* K *is some positive constant.*

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[1] In the sequel (′) will denote differentiation with respect to θ.

[2] It was suggested by Dr. Bernard D. Coleman that boundedness results, comparable to the maximum and minimum principles associated with the linear heat equation, should follow from the thermodynamical assumptions that $\hat{\epsilon}'$ is positive and $\hat{\mathbf{q}}$ satisfies (1.2).

We note that the constant function $\theta(\mathbf{X}) \equiv \theta_0$ for all $\mathbf{X} \in \bar{\Omega}$ is the unique equilibrium temperature field satisfying

$$\operatorname{div} \hat{\mathbf{q}}(\theta, \nabla\theta) + \hat{r}_{\theta_0}(\theta) = 0, \qquad \mathbf{X} \in \Omega, \tag{1.8a}$$

$$\theta(\mathbf{X}) = \theta_0, \qquad \mathbf{X} \in \partial\Omega. \tag{1.8b}$$

THEOREM 2. *Let $\hat{\epsilon}$ and $\hat{\mathbf{q}}$ satisfy Assumption* 1 *and let f be any $C^2(\bar{\Omega})$ function which satisfies*

$$\hat{\mathbf{q}}(f(\mathbf{X}), \nabla f(\mathbf{X})) \cdot \mathbf{n}(\mathbf{X}) = 0, \qquad \mathbf{X} \in \partial\Omega. \tag{1.9}$$ [3]

Define the number $\theta_0 > 0$ by

$$\theta_0 = \hat{\epsilon}^{-1}\left(\frac{\int_\Omega \hat{\epsilon}(f(\mathbf{X}))\, d\mathbf{X}}{mes(\Omega)}\right), \tag{1.10}$$

where $\hat{\epsilon}^{-1}$ is the inverse function of $\hat{\epsilon}$ and $mes(\Omega)$ is the volume of the domain Ω, and assume that in addition to its other properties f satisfies

$$0 < f(\mathbf{X}) < 2\theta_0, \qquad \mathbf{X} \in \bar{\Omega}. \tag{1.11}$$

If θ is any $C^2(\bar{\Omega} \times [0, \infty))$ function which satisfies (1.5), *the initial conditions that $\theta(\mathbf{X}, 0) = f(\mathbf{X})$ for all $X \in \bar{\Omega}$, and the boundary conditions that $\hat{\mathbf{q}}(\theta(\mathbf{X}, t), \nabla\theta(\mathbf{X}, t)) \cdot \mathbf{n}(\mathbf{X}) = 0$ for all $(\mathbf{X}, t) \in \partial\Omega \times (0, \infty)$, then, for all $t \geqslant 0$,*

$$\sup_{\mathbf{X}\in\bar{\Omega}} | \theta(\mathbf{X}, t) - \theta_0 | \leqslant V(f, \theta_0)\, e^{-Kt}, \tag{1.12}$$

where again $V(f, \theta_0) = \sup_{\mathbf{X}\in\bar{\Omega}} | f(\mathbf{X}) - \theta_0 |$ and K is some positive constant.

We mention that the constant function $\theta(\mathbf{X}) \equiv \theta_0$ for all $\mathbf{X} \in \bar{\Omega}$ is the unique equilibrium temperature field satisfying

$$\operatorname{div} \hat{\mathbf{q}}(\theta, \nabla\theta) + \hat{r}_{\theta_0}(\theta) = 0, \qquad \mathbf{X} \in \Omega, \tag{1.13a}$$

$$\hat{\mathbf{q}}(\theta, \nabla\theta) \cdot \mathbf{n} = 0, \qquad \mathbf{X} \in \partial\Omega, \tag{1.13b}$$

and

$$\int_\Omega \epsilon(\theta(\mathbf{X}))\, d\mathbf{X} = \int_\Omega \epsilon(f(\mathbf{X}))\, dX. \tag{1.13c}$$

2. PROOF OF THEOREMS 1 AND 2

We shall only prove Theorem 1; the proof of Theorem 2 is similar. We first record some Remarks which will be of use in the sequel.

[3] $\mathbf{n}(\mathbf{X})$ is the outward normal to the surface $\partial\Omega$ at the point $\mathbf{X}$.

REMARK 2.1.

(a) Since f satisfies (1.6b), we may find a $\delta \in (0, \theta_0)$ such that

$$V(f, \theta_0) < \theta_0 - \delta \tag{2.1}$$

(b) If T is any positive number, δ any positive number satisfying (2.1), and θ any $C^2(\bar{\Omega} \times [0, \infty))$ function satisfying the hypotheses of the Theorem 1, then there exists a number $h(\delta, T) > 0$ such that

$$\sup_{(\mathbf{X},t)\in\bar{\Omega}\times[0,h(\delta,T)]} \theta(\mathbf{X}, t) \leqslant \theta_0 + V(f, \theta_0) + \delta < 2\theta_0 , \tag{2.2a}$$

$$\inf_{(\mathbf{X},t)\in\bar{\Omega}\times[0,h(\delta,T)]} \theta(\mathbf{X}, t) \geqslant \theta_0 - V(f, \theta_0) - \delta > 0, \tag{2.2b}$$

and

$$| \theta(\mathbf{X}, t) - \theta(\mathbf{X}, \tau)| \leqslant \delta \tag{2.2c}$$

for all $\mathbf{X} \in \bar{\Omega}$ and $(t, \tau) \in [0, T]$ satisfying $| t - \tau | \leqslant h(\delta, T)$.

(c) Let T, δ, and θ be as in part (b) of the remark, and let $\hat{\epsilon}$ satisfy Assumption 1 (a). We may then conclude that there exist positive numbers C_{δ,θ_0} and M_{δ,θ_0}, depending only on θ_0 and δ, such that

$$\hat{\epsilon}'(\theta(\mathbf{X}, t)) \leqslant C_{\delta,\theta_0} \tag{2.3a}$$

and

$$\left| \frac{(\theta(\mathbf{X}, t) - \theta_0)\, \hat{\epsilon}''(\theta(\mathbf{X}, t))}{\hat{\epsilon}'(\theta(\mathbf{X}, t))} \right| \leqslant M_{\delta,\theta_0} \tag{2.3b}$$

for all $(\mathbf{X}, t) \in \bar{\Omega} \times [0, h(\delta, T)]$.

PROOF. The remark is a consequence of the smoothness of the function θ, the fact that $\theta(\mathbf{X}, 0) = f(\mathbf{X})$ for all $\mathbf{X} \in \bar{\Omega}$, and the properties of f.

REMARK 2.2.

(a) If g is in $C(\bar{\Omega})$, then

$$\sup_{\mathbf{X}\in\bar{\Omega}} | g(\mathbf{X})| = \lim_{n\to\infty} \left(\int_{\Omega} g(\mathbf{X})^{2n}\, d\mathbf{X}\right)^{1/2n} \tag{2.4}$$

(b) If g satisfies the inequality

$$\frac{dg}{dt} \leqslant -Kg, \qquad K > 0, \tag{2.5}$$

then g satisfies the inequality

$$g(t) \leqslant g(0)\, e^{-Kt}. \tag{2.6}$$

PROOF. A proof of part (a) may be found in [3, p. 150]; part (b) is immediately evident.

We now prove Theorem 1. If we let T, δ, and $h(\delta, T)$ be as in Remark 2.1 (b) and define the functional $I_{2n}(\theta(\cdot, t) - \theta_0)$ by

$$I_{2n}(\theta(\cdot, t) - \theta_0) = \int_\Omega (\theta(\mathbf{X}, t) - \theta_0)^{2n} \, d\mathbf{X}, \tag{2.7}$$

then, for functions θ satisfying the hypotheses of Theorem 1, $I_{2n}(\theta(\cdot, t) - \theta_0)$ is $C^1[0, \infty)$ and its derivative is given by

$$\frac{dI_{2n}}{dt}(\theta(\cdot, t) - \theta_0) = 2n \int_\Omega (\theta(\mathbf{X}, t) - \theta_0)^{2n-1} \frac{\partial \theta}{\partial t}(\mathbf{X}, t) \, d\mathbf{X}. \tag{2.8}$$

If we restrict our attention to the set $\bar{\Omega} \times [0, h(\delta, T)]$, then

$$\frac{1}{\hat{\epsilon}'(\theta(\mathbf{X}, t))} \geqslant \frac{1}{C_{\delta, \theta_0}} > 0,$$

and hence Eq. (1.5), the boundary conditions, and the divergence theorem imply that

$$\begin{aligned} \frac{dI_{2n}}{dt}(\theta(\cdot, t) - \theta_0) = & -2n \int_\Omega \frac{(\theta(\mathbf{X}, t) - \theta_0)^{2n-2}}{\hat{\epsilon}'(\theta(\mathbf{X}, t))} [\hat{r}_{\theta_0}(\theta(\mathbf{X}, t))(\theta(\mathbf{X}, t) - \theta_0)] \, d\mathbf{X} \\ & + 2n \int_\Omega \frac{(\theta(\mathbf{X}, t) - \theta_0)^{2n-2}}{\hat{\epsilon}'(\theta(\mathbf{X}, t))} \hat{\mathbf{q}}(\theta(\mathbf{X}, t), \nabla\theta(\mathbf{X}, t)) \\ & \cdot \nabla\theta(\mathbf{X}, t) \left\{ (2n - 1) - \frac{(\theta(\mathbf{X}, t) - \theta_0) \, \hat{\epsilon}''(\theta(\mathbf{X}, t))}{\hat{\epsilon}'(\theta(\mathbf{X}, t))} \right\} d\mathbf{X}. \end{aligned} \tag{2.9}$$

It now follows from equations (1.2), (1.4), (2.3a), (2.3b), and (2.9) that for all times $t \in [0, h(\delta, T)]$ and all integers $n > (M_{\delta, \theta_0} + 1)/2$

$$\frac{dI_{2n}}{dt}(\theta(\cdot, t) - \theta_0) \leqslant \frac{-2nr_0}{C_{\delta, \theta_0}} I_{2n}(\theta(\cdot, t) - \theta_0). \tag{2.10}$$

Equation (2.10) and Remark 2.2 (b) allow us to conclude that for all $t \in [0, h(\delta, T)]$

$$I_{2n}(\theta(\cdot, t) - \theta_0) \leqslant \left\{ \int_\Omega (f(\mathbf{X}) - \theta_0)^{2n} \right\} \exp\left(\frac{-2nr_0}{C_{\delta, \theta}} t \right). \tag{2.11}$$

Eq. (2.11) and Remark 2.2 (a) then imply that for all $t \in [0, h(\delta, T)]$

$$\sup_{\mathbf{X} \in \bar{\Omega}} | \theta(\mathbf{X}, t) - \theta_0 | \leqslant V(f, \theta_0) \exp\left(\frac{-r_0}{C_{\delta, \theta_0}} t \right). \tag{2.12}$$

It now follows from (2.12) and (2.2c) that we may extend (2.3) and hence (2.10), (2.11), and (2.12) to the whole interval $[0, T]$. Since the number $T > 0$ was arbitrary, the proof of Theorem 1 is complete.

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The author and Dr. Coleman are currently studying the dynamical stability properties of Gibbs' stable equilibrium states for more general thermodynamical systems and would like to point out that the results of Theorems 1 and 2 are refinements of theorems which apply to a wider range of thermodynamical systems.

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